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Highlights

- A dynamic adaptive group AHP consensus reaching model.
- Prioritizing decision makers based on a Markov chain method.
- A peer to peer opinion comparison and exchange method.
- An automatic feedback mechanism as the engine of the dynamic adaptive consensus model.

A peer-to-peer dynamic adaptive consensus reaching model for the group AHP decision making

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Abstract: Consensus reaching models are widely applied in group decision making problems to improve the group's consensus level before making a common decision. Within the context of the group Analytic Hierarchy Process (AHP), a novel consensus reaching model in a dynamic decision environment is proposed. A Markov chain method can be used to determine the decision makers' weights of importance for the aggregation process with respect to the group members' opinion transition probabilities. The proposed group consensus reaching model facilitates a peer to peer opinion exchange process which relieves the group of the need for a moderator by using an automatic feedback mechanism. Moreover, as the elements in the group decision framework change in a dynamic decision making problem, this model provides feedback suggestions that adaptively adjust for each of the decision makers depending on his credibility in each round. The full process of the dynamic adaptive consensus reaching model is presented and its properties are discussed. Finally, a numerical example is given to demonstrate the effectiveness of our model.

Keywords: Group decision making; Consensus reaching; weight determining; The Analytic Hierarchy Process (AHP)

1 Introduction

The complexity of the decisions that management face makes it difficult to depend on a single decision maker's knowledge and capabilities to obtain a meaningful and reliable solution. Therefore, group decision making has received significant attention in both the research and in practice. Group decision making (GDM) is a procedure that combines the individuals' judgments into a common opinion on behalf of a whole group. To express the judgments of individuals, several formats are usually used in GDM, such as fuzzy preference relations (Cabrerizo, Moreno, et al., 2010; Tanino, 1984; Y. Xu, et al., 2013; Z. Xu, 2009), linguistic preference relations (Alonso, et al., 2013; Herrera, et al., 1995; Herrera, et al., 1996; Wu and Xu, 2012a), utility functions (Brock, 1980; Greco, et al., 2012; Huang, et al., 2013; Keeney and Kirkwood, 1975) and the Analytic Hierarchy Process (AHP) (Altuzarra, et al., 2010; Chiclana, et al., 2001; Dyer and Forman, 1992; Van Den Honert and Lootsma, 1997).

The Analytic Hierarchy Process (AHP), which is a comprehensive tool developed by Saaty (1977) for constructing decision models and establishing the decision priorities with respect to a finite set of alternatives, has been widely applied to group decisions because of the flexible structure and our innate ability to make relative comparisons. Allocating the weight or importance to each individual within a group is an important component in the decision process and plays a key role in obtaining the final solution in an AHP model. In the past three decades multiple methods have been proposed to determine the weights of individuals (Bolloju, 2001; Forman and Peniwati, 1998; Ramanathan and Ganesh, 1994; Saaty, 1994a; Van den Honert, 2001). However, these methods suffer from several drawbacks. First, most of these methods assign the weights according to subjective judgments. Thus at least one individual must serve as a judge of the judges to provide this subjective weighting for the preferences of the decision makers. In practice, this potential for bias is a significant obstacle to overcome. Furthermore, it could be more reasonable to assign the weights of importance to each decision maker according to how compatible their judgments are with those of others

(Y. Xu, et al., 2013; Z. Xu and Cai, 2011). Therefore we develop a dynamic method using the opinion transition probabilities, which serve as a way to measure the compatibility between decision makers, to allocate the weights to the decision makers in place of needing a judge.

As there is always a diversity of opinion in a group, reaching a certain level of consensus is a critical step in obtaining a valid solution in real world group decision making problems (Herrera, et al., 1996). However, full consensus, which is related to the state of total agreement, is hard to achieve in real world GDM problems. Therefore the concept of soft consensus is employed in GDM problems (Herrera-Viedma, et al., 2014). A consensus reaching process is usually defined as an interactive process with several rounds of improving the incompatible decision maker's consensus level. Numerous approaches have been developed for measuring and improving the consensus based on different preference relations, such as linguistic consensus reaching models (Y. Dong, Xu, et al., 2010; Herrera-Viedma, et al., 2005; Herrera and Herrera-Viedma, 2000; Mata, et al., 2009), fuzzy consensus reaching models (Cabrerizo, Perez, et al., 2010; Guha and Chakraborty, 2011; Kacprzyk, et al., 1992; Parreiras, et al., 2010), and evidential reasoning based consensus models (Fu and Yang, 2010, 2011, 2012).

Consensus reaching models have also been widely studied in group AHP decision making problems. Using consistency as a control in group decision making, Y. Dong, Zhang, et al. (2010) proposed two AHP consensus models based on a row geometric mean prioritization method. Wu and Xu (2012b) developed a consistency and consensus based group AHP decision making model which is independent of the method of prioritization. Gong, et al. (2012) proposed a group consensus deviation degree optimization model for the group AHP problems and proved that the consensus degree of decision makers converges as the number of the decision makers increases indefinitely. Y. Xu, et al. (2013) presented a distance-based consensus model for group AHP applications, in which the individual to group consensus index and group consensus index are introduced in an iterative algorithm for consensus reaching. However, as pointed out in Section 4, a significant drawback that exists in their consensus reaching models is that they use the aggregated group opinion as the reference point for both measuring an individual's consensus level and revising the incompatible individual's judgments. These centricity-oriented methods require aggregating the individuals' opinions into a representative group opinion in each round. This step increases the cost and complexity of the GDM problems. Thus a new consensus reaching model without the need for opinion aggregation in each round is desirable to avoid this drawback. Furthermore, in existing consensus reaching models, it is often the case that they behave in a similar way during the whole consensus process even though the elements and conditions of the GDM problem change. Mata, et al. (2009) proposed an adaptive consensus reaching model which adapts the number of changes required by the experts according to the consensus level achieved in each round of the consensus reaching process. Chen, et al. (2012) proposed an adaptive consensus support model which can modify experts' preferences to improve convergence toward a higher degree of consensus. In a peer to peer consensus reaching model, it is the authors' belief that how much a decision maker should revise her priorities in each round of the consensus reaching process might be determined by her consensus level or credibility in that round. A new consensus reaching process can address these issues within GDM.

Thus we first use a finite state space Markov chain to develop an opinion transition probabilities based weighting method. Then a peer to peer dynamic adaptive consensus reaching model for the group AHP decision problems is proposed. The automatic feedback mechanism of the consensus reaching model will adapt the portion of the decision maker's judgments that are kept and/or revised according to their credibility in a dynamic group consensus reaching process.

The paper is organized as follows: Section 2 displays some preliminaries related to GDM problems. Section 3 presents the objective weight determining method. Section 4 provides the detailed description of the proposed peer to peer dynamic adaptive consensus model in AHP-group decision making. A numerical example is provided in Section 5. Finally, some concluding remarks are presented in Section 6.

2 Preliminaries

For simplicity, we use $N = \{1,2,\cdots,n\}$, $M = \{1,2,\cdots,m\}$ to denote the elements in sets. Let $X = \{x_1, x_2, \cdots, x_n\}$ be a finite set of alternatives, where x_i denotes the *i*th alternative. The judgment information is represented as an $n \times n$ pairwise compare matrix (PCM) $\boldsymbol{A} = \left(a_{ij} = w_i / w_j\right)_{n\times n}$, where $a_{ij} = 1/a_{ji}$ and a_{ij} belongs to Saaty's 1-9 fundamental scale and represents the relative importance or better, dominance of x_i over x_j. Consistency, defined by Saaty (1980), is a concept defined to describe and reflect the quality of a PCM. The PCM $\boldsymbol{A}$ is perfectly consistent if

$$a_{ij} = a_{ik} a_{kj}, \quad i, j, k \in N \tag{1}$$

However, in real life decision situations, consistency is hard to achieve. Saaty (1980) defined the consistency index as

$$CI_{\boldsymbol{A}} = \frac{\lambda_{\max} - n}{n - 1} \tag{2}$$

where $\lambda_{\max}$ is the largest or principal eigenvalue of $\boldsymbol{A}$. To measure the inconsistency of $\boldsymbol{A}$, we use the consistency ratio (Saaty, 1977)

$$CR_{\boldsymbol{A}} = \frac{CI_{\boldsymbol{A}}}{RI_n} \tag{3}$$

where RI_n is average random consistency index derived from randomly generated $n \times n$ PCMs (see Table 1). In general, if $CR_{\boldsymbol{A}}$ is less than 0.10, we say that PCM $\boldsymbol{A}$ is acceptably consistent(Saaty, 1990).

Table 1. The random consistency index. (Saaty, 2010)

n	1	2	3	4	5	6	7	8	9	10
RI	0	0	0.52	0.89	1.11	1.25	1.35	1.40	1.45	1.49

Let $D = \left\{DM_1, DM_2, \cdots, DM_m\right\}$ be a set of m decision makers. The judgment of decision maker DM_k is expressed by a PCM $\boldsymbol{A}_k = \left(a_{ij(k)}\right)_{n\times n}$, for $k \in M$, and let $\boldsymbol{\rho} = (\rho_1, \rho_2, \cdots, \rho_m)^{\mathrm{T}}$ be the weight or importance vector of the decision makers, where

$\rho_k \geq 0$, $\sum_{k=1}^{m} \rho_k = 1$, $k \in M$. By aggregating with the weighted geometric mean, the group PCM $\boldsymbol{G} = (g_{ij})_{n\times n}$ can be calculated as

$$g_{ij} = \prod_{k=1}^{m} (a_{ij(k)})^{\rho_k} \tag{4}$$

Once the individual judgments are aggregated by the geometric mean into group judgments, we face the following question for the inconsistency of the group PCM. Grošelj and Zadnik Stirn (2012) proved that if each of m PCMs $\boldsymbol{A}_1, \boldsymbol{A}_2, \cdots, \boldsymbol{A}_m$ is of acceptable consistency, then $\boldsymbol{G}$ (the weighted geometric mean of these PCMs) is also of acceptable consistency.

To measure the consensus level in a group, one first measures the closeness or distance of opinions of two decision makers. In the group AHP context, it is necessary to compare the difference, closeness, or distance of two PCMs. Then using these distance measures which are calculated between all pairs of decision makers in a group, we can create an undirected graph to show the relationship of the decision makers. Considering that a PCM belongs to an absolute scale and thus also to a ratio scale, Saaty (1994b) suggested that the closeness of two PCMs can be measured by using the compatibility index.

Definition 1 (Saaty, 1994b) Let $\boldsymbol{A}_k = (a_{ij(k)})_{n\times n}$ and $\boldsymbol{A}_l = (a_{ij(l)})_{n\times n}$ be two PCM, the compatibility index of $\boldsymbol{A}_k$ and $\boldsymbol{A}_l$ can be defined as

$$c(\boldsymbol{A}_k, \boldsymbol{A}_l) = \frac{1}{n^2} \boldsymbol{e}^{\mathrm{T}} \boldsymbol{A}_k \circ \boldsymbol{A}_l^{\mathrm{T}} \boldsymbol{e} = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij(k)} a_{ji(l)} \tag{5}$$

where $\circ$ denotes the Hadamard product of two matrices and $\boldsymbol{e} = (1, 1, \cdots, 1)^{\mathrm{T}}$. From Eq. (5), we have $c(\boldsymbol{A}_k, \boldsymbol{A}_l) \geq 1$ and $c(\boldsymbol{A}_k, \boldsymbol{A}_l) = 1$ if and only if $\boldsymbol{A}_k = \boldsymbol{A}_l$. Then we can define the individual consensus index.

Definition 2 Let $\boldsymbol{A}_1, \boldsymbol{A}_2, \cdots \boldsymbol{A}_m$ and $\boldsymbol{G}$ be as before, and $\boldsymbol{A}_p$, $\boldsymbol{A}_q$ be two PCMs with respect to decision maker DM_p, DM_q, $p, q \in M$, the individual consensus index

(ICI) between DM_p and DM_q is defined as

$$ICI_{pq} = c(\boldsymbol{A}_p, \boldsymbol{A}_q), \quad p, q \in M \tag{6}$$

where ICI_{pq} denotes the closeness between the judgments of decision maker DM_p and DM_q in the group. From Eq.(5) and Eq.(6), we have that the ICI holds: (1) $ICI_{pq} \geq 1$; (2) $ICI_{pq} = ICI_{qp}$ (symmetry); (3) $ICI_{pp} = 1$ (reflexivity).

Given a threshold value ε, where $\varepsilon \geq 1$, if $ICI_{pq} \leq \varepsilon$, we say that decision maker p and q have acceptable consensus amongst each other. The value of ε usually depends on the particular problem we are dealing with. Saaty (1994b) suggested that ε can be set at 1.1 as a lower consensus level. From long experience with projects we have learned 10% is too large, and hence we suggest that ε can be set at 1.01 because 1% deviation is usually used as the upper end of acceptability. When the consequences of the decision are associated to every individual in this group and no one has unique authority, the consensus level should be as high as possible, such as 1.01. At the other extreme, if there is a leader in a group and the decision is made in an autocratic way, or it is urgent to get a solution to the problem, then a lower consensus level such as 1.1 could be required.

The individual consensus index can only denote the undirected relation between a pair of decision makers. In other words, we have $ICI_{pq} = ICI_{qp}$. Suppose we have a pair of individuals $\{DM_k, DM_l\}$ in a group, it is necessary to distinguish the adoption directions between these two individuals in the consensus reaching process because DM_k and DM_l may make a different choice about the judgment revision. Thus for clarifying the direction between two decision makers, we define the ordered pair of decision makers as follows:

Definition 3 Let $D = \{DM_1, DM_2, \cdots, DM_m\}$ be a set of m decision makers. An ordered pair of decision makers in the group is defined as DM_k, DM_l , $k, l \in M$, where, for example, DM_k, DM_l denotes that DM_k may revise his\her judgments according to DM_l.

From **Definition 3,** we get that $DM_k, DM_l \neq DM_l, DM_k$. Thus with the different ordered pairs of decision makers, the relations between one pair of decision makers can be shown as Figure 1.

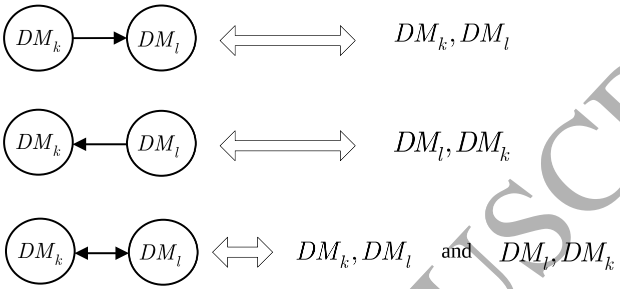

Figure 1. Directed relations in a pair of decision makers

3 Determining the weight of decision makers in a group

The weight or importance of a decision maker is an important and sensitive issue in group decision making. As often is the case, the weight of the decision makers can be obtained through pairwise comparisons or by rating them one at a time. Saaty (1994a) proposed a hierarchy of criteria such as expertise, experience, previous performance, persuasive abilities, effort on the problem, etc. to determine the weight vector of the decision makers, as shown in Figure 2. This is a natural way to determine the individual weight in a group by using prior information about the decision makers. Ramanathan and Ganesh (1994) provided an eigenvector based method to derive the weight vector of decision makers by using interpersonal comparison among individuals in a group.

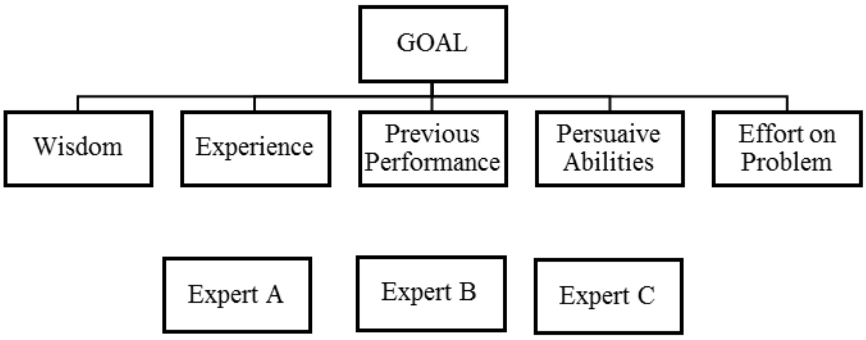

Figure 2. Hierarchy for judging experts (Saaty, 1994a)

Forman and Peniwati (1998) noted that in some cases it is difficult to find a knowledgeable person to provide judgments to the hierarchy in Figure 2. In general, even if one does find such an individual, he may not have the knowledge and information to determine which decision maker is more important. In addition, the decision makers may not know each other very well. Thus, in the absence of prior information, an objective way to determine the weights of the decision makers as shown below is needed.

Usually, the objective way to determine decision makers' weights is to find an appropriate weight vector which can aggregate all the individual judgments into the most representative group judgment. In many instances an optimization program that minimizes the distance between all of the decision makers' opinions and the aggregated group opinion is used (Y. Xu, et al., 2013; Z. Xu and Cai, 2011). However, this optimization programming relies on aggregating the individuals' opinions into the group's opinion. Here we propose to determine the weights of the decision makers based on their opinion transition probabilities. In a group of decision makers, there is always a diversity of opinion. However, decision maker DM_k may have a similar opinion with decision maker DM_j and have very different opinion from decision maker DM_l, $j,k,l \in M$. The differences in the proximity between the decision makers reveal the possibilities of opinion transitions from one decision maker to other decision makers. Generally, it is easier to have DM_k change his opinion to the same as or more similar to DM_j's opinion than to DM_l's because of the proximity of their respective opinions. Also the opinion transition possibility shows the level of agreement between one decision maker and another. This opinion transition process can be seen as a finite state space Markov chain. Considering we defined the ICI index to measure the closeness of two decision makers' judgment in a group AHP context, we can define the opinion transition probability from one decision maker to another:

$$p_{vs} = \frac{1/ICI_{vs}}{\sum_{s=1}^{m} \frac{1}{ICI_{vs}}}, \quad v \in M \tag{7}$$

where $\sum_{s}^{m} p_{vs} = 1$, $p_{vs} > 0$. Then by calculating the opinion transition probabilities between any two decision makers in D, we have a Markov matrix of opinion transitions $\boldsymbol{P} = \left(p_{vs}\right)_{m\times m}$, which is irreducible and ergodic because all of the elements in this matrix are positive (Kijima, 1997). The limiting distribution which is also the only stationary distribution of this Markov chain can be seen as the power or weight distribution of decision makers in a group. The limiting distribution $\boldsymbol{\pi}$ can be calculated by (Ross, 2009):

$$\boldsymbol{\pi}\boldsymbol{P} = \boldsymbol{\pi} \tag{8}$$

where $\boldsymbol{\pi} = (\pi_1, \pi_2, \cdots, \pi_m)^{\mathrm{T}}$, $\sum_{k}^{m} \pi_k = 1$, $\pi_k \geq 0$. Let $\rho_k = \pi_k$, we can get the weight vector $\boldsymbol{\rho} = (\rho_1, \rho_2, \cdots, \rho_m)^{\mathrm{T}}$ of decision makers.

Remark 1. The proposed weight allocating method relies on neither a judge's subjective judgment nor the distance to the aggregated group opinion, but rather on the comparison among the individuals. The advantage of our method over the subjective ones is that the subjective judgment from the judge of judges is difficult to obtain in the real application. Also the subjective bias of a judge will always exist and can be difficult to reduce. Previously, the objective way to find an optimized weight vector was associated with the aggregated group opinion. Thus the distance or similarity between the individuals to the aggregated group is used in these centricity-oriented methods. However, the proximity among the individuals, which contains more information, has not been used in prior methods. Thus our method, by using this peer to peer information to avoid this loss of information can provide more persuasive and reliable results. Moreover, a precondition of the centricity oriented methods is to define or find an appropriate aggregation function. Different aggregation functions may cause different results of weight allocating. Thus the robustness of such weight determining method is weaker than the proposed method.

4 The peer-to-peer consensus reaching model

As a certain level of group consensus is a necessity in group decision making, during the group decision making process, consensus reaching models can be applied to aid the decision makers to improve their consensus level. In a traditional consensus reaching model, there is often a moderator, via the collection and exchange of opinions, who tries to give advice to the experts on how to revise or update their opinions to bring their judgments closer together. As a result, subjectivity may be introduced to the consensus reaching process by the moderator. Thus as shown in Figure 3, we give a group AHP consensus reaching model incorporating a feedback mechanism as a substitute for the moderator's suggestions.

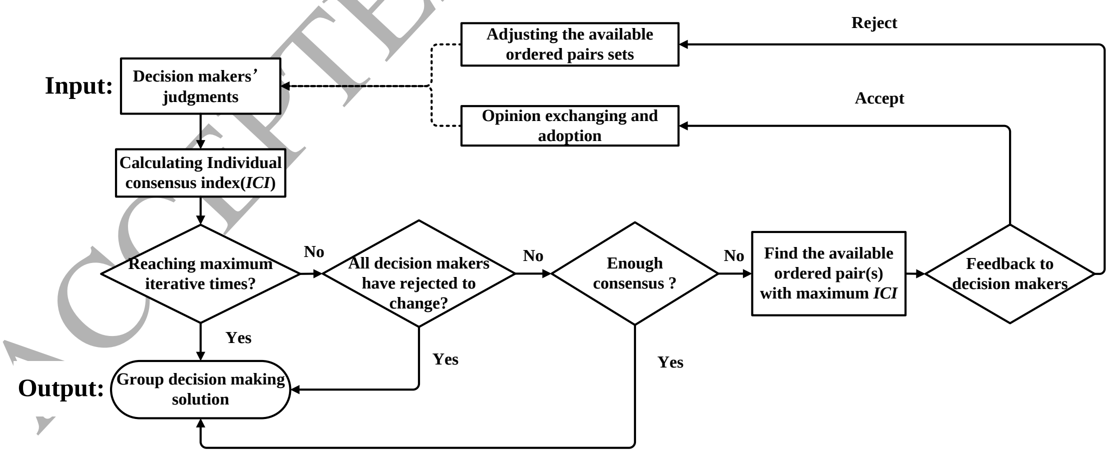

Figure 3 Diagram of the peer-to-peer dynamic adaptive consensus reaching process

The feedback mechanism, which determines how to improve the consensus level, is crucial in a group consensus reaching model. Without loss of generality, suppose that

DM_a and DM_b are selected in the tth iteration, $a, b \in M$, feedback suggestions should be provided to help them revise their PCMs and improve their consensus level. The new PCMs $\boldsymbol{A}_a^{t+1}$ and $\boldsymbol{A}_b^{t+1}$ should be constructed by using the following strategy:

$$a_{ij(a)}^{t+1} = {a_{ij(a)}^{t}}^{\alpha_a^t} \; {a_{ij(b)}^{t}}^{1-\alpha_a^t} \tag{9}$$

$$a_{ij(b)}^{t+1} = {a_{ij(b)}^{t}}^{\alpha_b^t} \; {a_{ij(a)}^{t}}^{1-\alpha_b^t} \tag{10}$$

where α_τ^t is a parameter which determines what amounts and relative portion of the new PCM $\boldsymbol{A}_\tau^{t+1}$ of decision maker DM_τ is from his own original PCM $\boldsymbol{A}_\tau^t$. As a dynamic adaptive consensus reaching model, the portion of the experts' prior judgments that are preserved should be updated in each round of the consensus reaching process, so that as the consensus level increases the portion of the judgments preserved should also increase. In addition, as can be seen in Eq.(9) and Eq.(10), the new PCM of a selected decision maker is composed of his/her original PCM and the original PCM from another selected decision maker, who is the most incompatible one with him in the group. Thus determining the portion of judgment preservation is also determining which decision maker is more credible. It is reasonable that decision makers with higher incompatibility with other decision makers will need more guidance than those decision makers that with higher consensus levels. Thus when the dynamic adaptive consensus reaching model advises the decision makers to revise their judgments, the amount of judgment preservation required for each decision maker is adjusted *depending on his degree of consensus with the unselected decision makers*; and have

$$\alpha_a^t = 1 - \frac{\sum_{l=1, l \neq a,b}^{m} ICI_{al}}{2\left(\sum_{l=1, l \neq a,b}^{m} ICI_{al} + \sum_{l=1, l \neq a,b}^{m} ICI_{bl}\right)} \tag{11}$$

$$\alpha_b^t = 1 - \frac{\sum_{l=1, l \neq a,b}^{m} ICI_{bl}}{2(\sum_{l=1, l \neq a,b}^{m} ICI_{al} + \sum_{l=1, l \neq a,b}^{m} ICI_{bl})} \tag{12}$$

From Eq.(11) and Eq.(12), 3 important properties are worth addressing: (1) $\alpha_\tau^t \in (0.5,1)$, $\tau \in \{a,b\}$. Thus α_τ^t is bounded to make sure to avoid too large of a change in the decision makers' opinion which helps preserve their sovereignty and the value of their input. (2) if $\sum_{l=1,l\neq a,b}^{m} ICI_{al} \geq \sum_{l=1,l\neq a,b}^{m} ICI_{bl}$, then $\alpha_a^t \leq \alpha_b^t$, and vice versa. It shows that if one of two selected decision makers has a lower degree of consensus with the unselected decision makers than another, therefore he\she has lower credibility and should be required to make a greater change to his\her judgments. (3) $\alpha_a^t = \alpha_b^t$ if and only if $\sum_{l=1,l\neq a,b}^{m} ICI_{al} = \sum_{l=1,l\neq a,b}^{m} ICI_{bl}$. This property implies that if they have the same consensus level with unselected decision makers, they have the same degree of credibility.

By using this dynamic and adaptive strategy, the proposed peer to peer dynamic adaptive consensus model will aid all the decision makers to reach a predefined level of acceptable consensus. The details of our peer to peer dynamic adaptive consensus model are depicted in the following Algorithm.

Input: Initial PCMs $\boldsymbol{A}_1, \boldsymbol{A}_2 \cdots, \boldsymbol{A}_m$ with acceptable levels of consistency according to the threshold value of the individual consensus index ε, and the maximum number of iterations T.

Output: Final PCMs $\boldsymbol{A}_1^*, \boldsymbol{A}_2^*, \cdots, \boldsymbol{A}_m^*$, group PCM $\boldsymbol{G}^*$, and the number of iterations t^*, $0 \leq t^* \leq T$.

Step 1. Let D^t be the set of all ordered pairs in which the first decision maker has not rejected the updating recommendation from the moderator before the t th iteration. Let $D^0 = \quad DM_1, DM_2 \ , \cdots, \ DM_1, DM_m \ , \cdots, \ DM_m, DM_1 \ , \cdots, \ DM_m, DM_{m-1} \quad , \ t = 0$,

$\boldsymbol{A}_k^0 = \left(a_{ij(k)}^0\right)_{n\times n} = \left(a_{ij(k)}\right)_{n\times n}$.

Step 2. Calculate the individual consensus index ICI_{kl}^t for all $(DM_k, DM_l) \in D^t$. If (1) $t \geq T$, or (2) $D^t = \varnothing$, or (3) $\forall (DM_k, DM_l) \in D^t$, $ICI_{kl}^t \leq \varepsilon$, then go to **Step 4**; otherwise, continue.

Step 3. Identify the most incompatible decision makers DM_p and DM_q with $ICI_{pq}^t = \max_{D^t} \left\{ICI_{kl}^t\right\}$. At least one of the ordered pairs (DM_p, DM_q) and (DM_q, DM_p) should be in D^t. As such we have 3 cases:

(1) Both (DM_p, DM_q) and (DM_q, DM_p) are in D^t. When both DM_p and DM_q to update their PCMs by using Eq.(9) and Eq.(10). We have 4 possible scenarios:

(a) If both of them accept this feedback suggestion, we have $\boldsymbol{A}_k^{t+1} = \left(a_{ij(k)}^{t+1}\right)_{n\times n}$, $k \in M$, where

$$a_{ij(k)}^{t+1} = \begin{cases} \left(a_{ij(k)}^t\right)^{\alpha_k^t} \left(a_{ij(q)}^t\right)^{1-\alpha_k^t}, & k = p \\ \left(a_{ij(k)}^t\right)^{\alpha_k^t} \left(a_{ij(p)}^t\right)^{1-\alpha_k^t}, & k = q \\ a_{ij(k)}^t, & k \neq p, q \end{cases} \tag{13}$$

And then set $t = t+1$, $D^{t+1} = D^t$, and return to **Step 2**.

(b) If DM_x rejects and DM_y accepts this feedback suggestion, $x, y \in \{p, q\}$, $x \neq y$, we have $\boldsymbol{A}_k^{t+1} = \left(a_{ij(k)}^{t+1}\right)_{n\times n}$, $k \in M$, where

$$a_{ij(k)}^{t+1} = \begin{cases} \left(a_{ij(y)}^t\right)^{\alpha_k^t} \left(a_{ij(x)}^t\right)^{1-\alpha_k^t}, & k = y \\ a_{ij(k)}^t, & k \neq y \end{cases} \tag{14}$$

Then set $t = t+1$, $D^{t+1} = D^t - (DM_x, DM_y)$, and return to **Step 2**.

(c) If both of them reject this feedback suggestion. Then set $\boldsymbol{A}_k^{t+1} = \boldsymbol{A}_k^t$, $k \in M$, $t = t+1$, $D^{t+1} = D^t - (DM_p, DM_q) - (DM_q, DM_p)$, and return to **Step 2**.

(2) DM_q, DM_p is in D^t and DM_p, DM_q is not. Feedback is only provided to DM_q according to Eq.(9). If DM_q rejects this suggestion, then set $t = t+1$ this directed pair is deleted from D^t. We now have $D^{t+1} = D^t - (DM_q, DM_p)$, and return to **Step 2**.

If DM_q accepts the feedback, then we have $\boldsymbol{A}_k^{t+1} = a_{ij(k)}^{t+1}{}_{n\times n}$ which is the same as Eq. (14), $k \in M$. Now set $t = t+1$, $D^{t+1} = D^t$, and return to **Step 2**.

(3) DM_p, DM_q is in D^t and DM_q, DM_p is not. The feedback mechanism will advise DM_p to update judgments as Eq. (9). Similarly, if DM_p rejects, we have $D^{t+1} = D^t - (DM_p, DM_q)$. Set $t = t+1$ and return to **Step 2**.

If DM_p accepts, then we have $\boldsymbol{A}_k^{t+1} = a_{ij(k)}^{t+1}{}_{n\times n}$ which is the same as Eq.(14), $k \in M$. Next, set $t = t+1$, $D^{t+1} = D^t$, and return to **Step 2**.

Step 4. This step is implemented when one of the three stopping conditions is met. Let $\boldsymbol{A}_k^* = \boldsymbol{A}_k^t$, and determine the weight vector $\boldsymbol{\rho} = (\rho_1, \rho_2, \cdots, \rho_m)^{\mathrm{T}}$ by using Eq. (7) and Eq. (8). Then we can get the group PCM $\boldsymbol{G}^*$ by using Eq. (4). The consensus reaching process is stopped and the final group priority vector is obtained from $\boldsymbol{G}^*$.

Remark 2. As can be seen from Figure 3, there are 3 stop conditions to avoid entering an infinite loop in the consensus reaching process. First, ε is fixed in advance and represents the necessary level of group consensus. If all decision makers meet the requirements of consensus, then the process should be terminated. However, if the value of ε is too high it may cause the consensus condition to never be satisfied, and as a consequence, we have an infinite looping consensus process. Thus in order to avoid this infinite loop, we define a parameter T in advance that limits the maximum number of iterations. Third, if all members in the group refuse to revise their judgment, the consensus reaching process should be ended.

Remark 3. A natural way to improve the consensus level in a group is to select the most incompatible decision makers because they have the biggest potential for consensus improvement. If the most incompatible pair of decision makers revise and come closer together then the upper bound of the entire group's incompatibility would be decreased. Based on this principle we develop the proposed consensus reaching algorithm and prove that this algorithm is convergent and effective. In our model, the feedback mechanism suggests only one pair of decision makers that has a maximum ICI in each round. If there are two or more pairs of decision makers with the same maximum ICI in the same round, they can also be given the feedback suggestion within that round.

As was discussed in Section 1, the decision makers are usually thought to follow the feedback suggestion to modify their judgments to support a successful group decision making process. However, this approach still gives them the opportunity to reject the suggestion without any penalties. In addition, the feedback mechanism can provide the suggestions automatically without a moderator. Furthermore, the additional burden on the decision makers through the subsequent rounds of the consensus reaching process is reduced. The decision makers only need to provide their initial judgments in the first round and then choose to follow the suggestion or not in the following steps of process. On the other hand, this method can be more interactive should the decision makers choose to compromise on the suggested feedback.

Remark 4. Previous studies on group consensus reaching models primarily use the aggregated group judgment as the reference point in the consensus reaching process (Perez, et al., 2014). They use this aggregated group judgments to find the incompatible decision makers and then revise their opinion using the group judgments as the model. In contrast to those centricity-oriented methods, the dynamic and adaptive process in the proposed peer to peer consensus reaching model always compares the information from one individual to another. As shown in Figure 3, this peer-to-peer dynamic adaptive consensus reaching model first calculates the ICI to find the most incompatible pair of decision makers. Then the automatic feedback mechanism provides the suggestion to each selected individual to update his\her original judgments to the new compromise judgments according to the opinion from the opposite side. In other words, no aggregated group judgments are used to serve as the reference points, nor are they used in the consensus improving process in our peer to peer consensus reaching model. This in turn reduces the costs and biases in the aggregation process.

The consistency of PCMs and the effectiveness of the consensus reaching process in our model is discussed below.

Theorem 1. In the proposed consensus reaching model, suppose all m initial PCMs $\boldsymbol{A}_1, \boldsymbol{A}_2, \cdots, \boldsymbol{A}_m$ provided by the decision makers are of acceptable consistency, then the final PCMs $\boldsymbol{A}_1^*, \boldsymbol{A}_2^*, \cdots, \boldsymbol{A}_m^*$ are of acceptable consistency.

Proof. The proof of **Theorem 1** is provided in Appendix A.

As the decision makers may change their PCMs according to the feedback suggestion, the acceptable consistency of the new PCMs should be checked and guaranteed. From Theorem 1, we know that the proposed consensus reaching process holds the consistent properties of the PCMs. It implies that if we can make sure the initial PCMs provided by the decision makers are of acceptable consistency, then we will have the acceptable consistent output after using our consensus reaching model. Therefore, we should do the consistency checks to make sure that all the input PCMs from the decision makers are all of acceptable consistency. In addition, the algorithm in our proposed consensus reaching model is convergent effective to improve the consensus level in a group.

Theorem 2. In the tth round of the proposed consensus reaching model, $0 \le t \le t^* - 1$, if $ICI_{kl}^t \ge \varepsilon$, $\forall k, l \in M$, then we have

$$\max_{k,l \in M} ICI_{kl}^{t+1} \le \max_{k,l \in M} ICI_{kl}^t \tag{15}$$

Proof. The proof of **Theorem 2** is provided in Appendix B.

Therefore, in each round of the consensus reaching process, the maximum incompatibility in a group will not exceed the maximum incompatibility in the last round. Thus the maximum individual consensus index should be non-increasing in our consensus reaching process. In the cases where the decision makers accept the modification provided by the automatic feedback mechanism, the proof shows that the model is convergent. Thus the feedback suggestion in our model is effective in increasing the level of group consensus.

5 Illustrative example

We use the following group decision making problem which was discussed by Y. Dong, Zhang, et al. (2010) , Wu and Xu (2012b) and Q. Dong and Saaty (2014). Suppose we have four alternatives X_1, X_2, X_3 and X_4 to be ranked and five decision makers DM_1, DM_2, DM_3, DM_4, and DM_5 with PCMs $\boldsymbol{A}_k = \left(a_{ij(k)}\right)_{4\times 4}$ and the corresponding priorities in the last column of the matrices, $k = 1,2,3,4,5$, where

$$\boldsymbol{A}_1 = \left(\begin{array}{cccc|c} 1 & 4 & 6 & 7 & 0.6168 \\ 1/4 & 1 & 3 & 4 & 0.2238 \\ 1/6 & 1/3 & 1 & 2 & 0.0972 \\ 1/7 & 1/4 & 1/2 & 1 & 0.0621 \end{array}\right), \quad \boldsymbol{A}_2 = \left(\begin{array}{cccc|c} 1 & 5 & 7 & 9 & 0.6526 \\ 1/5 & 1 & 4 & 6 & 0.2247 \\ 1/7 & 1/4 & 1 & 2 & 0.0762 \\ 1/9 & 1/6 & 1/2 & 1 & 0.0465 \end{array}\right),$$

$$\boldsymbol{A}_3 = \left(\begin{array}{cccc|c} 1 & 3 & 5 & 8 & 0.5705 \\ 1/3 & 1 & 4 & 5 & 0.2771 \\ 1/5 & 1/4 & 1 & 2 & 0.0959 \\ 1/8 & 1/5 & 1/2 & 1 & 0.0565 \end{array}\right), \quad \boldsymbol{A}_4 = \left(\begin{array}{cccc|c} 1 & 4 & 5 & 6 & 0.5970 \\ 1/4 & 1 & 3 & 3 & 0.2217 \\ 1/5 & 1/3 & 1 & 2 & 0.1084 \\ 1/6 & 1/3 & 1/2 & 1 & 0.0728 \end{array}\right),$$

$$\boldsymbol{A}_5 = \left(\begin{array}{cccc|c} 1 & 1/2 & 1 & 2 & 0.2185 \\ 2 & 1 & 2 & 3 & 0.4123 \\ 1 & 1/2 & 1 & 4 & 0.2679 \\ 1/2 & 1/3 & 1/4 & 1 & 0.1013 \end{array}\right).$$

The consistency ratios of $\boldsymbol{A}_k$ are: $CR_{\boldsymbol{A}_1} = 0.0383$, $CR_{\boldsymbol{A}_2} = 0.0678$,

$CR_{\boldsymbol{A}_3} = 0.0339$, $CR_{\boldsymbol{A}_4} = 0.0471$, $CR_{\boldsymbol{A}_5} = 0.0363$, which indicate that the given PCMs are of acceptable consistency. The threshold value of the individual consensus index is set such that $\varepsilon = 1.03$ and the maximum number of iterations $T = 10$. Now we show how to apply the proposed consensus reaching model to adjust the weights and update the judgments. Let $t = 0$, $\boldsymbol{A}_k^0 = \boldsymbol{A}_k$, $k = 1,2,3,4,5$, $D^0 = \quad DM_1, DM_2 \ ,\cdots,\ DM_1, DM_5 \ ,\cdots,\ DM_5, DM_1 \ ,\cdots\ DM_5, DM_4 \quad$. Using Eq.(6), we get the individual consensus indices matrix

$$\boldsymbol{O}^0 = \quad ICI_{kl}^0 \quad_{m\times m} = \begin{pmatrix} 1 & 1.0242 & 1.0167 & 1.0088 & 1.8017 \\ 1.0242 & 1 & 1.0268 & 1.0571 & \mathbf{2.0916} \\ 1.0167 & 1.0268 & 1 & 1.0323 & 1.6802 \\ 1.0088 & 1.0571 & 1.0323 & 1 & 1.7078 \\ 1.8017 & 2.0916 & 1.6802 & 1.7078 & 1 \end{pmatrix}$$

The threshold of the individual consensus level was not met. We have $ICI_{25}^0 = \max_{D^0} \quad ICI_{kl}^0$. Use Eq.(11) and Eq.(12), we have $\alpha_2^0 = 0.8127$, $\alpha_5^0 = 0.6873$. Then the automatic feedback mechanism will provide the revising suggestions to decision makers DM_2 and DM_5 as in Eq.(9) and Eq.(10). Suppose both of them accept the feedback suggestion, then we have

$$\boldsymbol{A}_2^1 = \left(\begin{array}{cccc|c} 1 & 13/4 & 34/7 & 34/5 & 0.5680 \\ 1/3 & 1 & 7/2 & 21/4 & 0.2702 \\ 1/5 & 2/7 & 1 & 16/7 & 0.1040 \\ 1/7 & 1/5 & 4/9 & 1 & 0.0578 \end{array}\right), \quad \boldsymbol{A}_5^1 = \left(\begin{array}{cccc|c} 1 & 1 & 11/6 & 16/5 & 0.3368 \\ 1 & 1 & 5/2 & 26/7 & 0.3752 \\ 1/2 & 2/5 & 1 & 29/9 & 0.2006 \\ 1/2 & 1/4 & 1/3 & 1 & 0.0873 \end{array}\right).$$

For the rest of decision makers, we have $\boldsymbol{A}_k^1 = \boldsymbol{A}_k^0$, $k = 1,3,4$. Set $D^1 = D^0$. Then we get

$$\boldsymbol{O}^1 = \quad ICI_{kl}^1 \quad_{m\times m} = \begin{pmatrix} 1.0000 & 1.0129 & 1.0167 & 1.0088 & \mathbf{1.2899} \\ 1.0129 & 1.0000 & 1.0044 & 1.0267 & 1.2162 \\ 1.0167 & 1.0044 & 1.0000 & 1.0323 & 1.2375 \\ 1.0088 & 1.0267 & 1.0323 & 1.0000 & 1.2476 \\ 1.2899 & 1.2162 & 1.2375 & 1.2476 & 1.0000 \end{pmatrix}$$

The threshold of the individual consensus level was still not met, $t = 1 < T$ and

$D^1 \neq \varnothing$. As can be seen from $\boldsymbol{O}^1$, $ICI_{15}^1 = \max_{D^1} ICI_{kl}^1$. We have $\alpha_1^1 = 0.7746$ and $\alpha_2^1 = 0.7254$. The feedback mechanism works as before. Suppose both DM_1 and DM_5 accept the feedback suggestion. We get $D^2 = D^1$ and

$$\boldsymbol{A}_1^2 = \left(\begin{array}{cccc|c} 1 & 3 & 23/5 & 47/8 & 0.5537 \\ 1/3 & 1 & 25/8 & 4 & 0.2591 \\ 2/9 & 1/3 & 1 & 20/9 & 0.1184 \\ 1/6 & 1/4 & 4/9 & 1 & 0.0687 \end{array}\right), \quad \boldsymbol{A}_5^2 = \left(\begin{array}{cccc|c} 1 & 3/2 & 3/2 & 4 & 0.4109 \\ 2/3 & 1 & 21/8 & 19/5 & 0.3366 \\ 2/5 & 3/8 & 1 & 17/6 & 0.1702 \\ 1/4 & 1/4 & 1/3 & 1 & 0.0822 \end{array}\right).$$

Then we have

$$\boldsymbol{O}^2 = \left(ICI_{kl}^2\right)_{m\times m} = \begin{pmatrix} 1.0000 & 1.0100 & 1.0177 & 1.0119 & 1.0664 \\ 1.0100 & 1.0000 & 1.0044 & 1.0267 & 1.1006 \\ 1.0177 & 1.0044 & 1.0000 & 1.0323 & 1.1171 \\ 1.0119 & 1.0267 & 1.0323 & 1.0000 & \mathbf{1.1186} \\ 1.0664 & 1.1006 & 1.1171 & 1.1186 & 1.0000 \end{pmatrix}$$

None of 3 stop conditions has been fulfilled yet. Next, the feedback suggestion is provided to DM_4 and DM_5. Suppose that DM_4 rejects the feedback suggestion. Set $D^3 = D^2 - (DM_4, DM_5)$ and have DM_5 update and we get

$$\boldsymbol{A}_5^3 = \left(\begin{array}{cccc|c} 1 & 18/9 & 3 & 22/5 & 0.4475 \\ 5/9 & 1 & 8/3 & 23/6 & 0.3171 \\ 1/3 & 3/8 & 1 & 8/3 & 0.1562 \\ 2/9 & 1/4 & 1/8 & 1 & 0.0792 \end{array}\right)$$

Then we have

$$\boldsymbol{O}^3 = \left(ICI_{kl}^3\right)_{m\times m} = \begin{pmatrix} 1.0000 & 1.0100 & 1.0177 & 1.0119 & 1.0360 \\ 1.0100 & 1.0000 & 1.0044 & 1.0267 & 1.0636 \\ 1.0177 & 1.0044 & 1.0000 & 1.0323 & \mathbf{1.0778} \\ 1.0119 & 1.0267 & 1.0323 & 1.0000 & - \\ 1.0360 & 1.0636 & 1.0778 & 1.0767 & 1.0000 \end{pmatrix}$$

The 3 stop conditions are not fulfilled. Thus $ICI_{35}^3 = \max_{D^3} ICI_{kl}^3$. Assume that both of these two decision makers accept the suggestion. Thus $D^4 = D^3$. We have

$$\boldsymbol{A}_3^4=\left(\begin{array}{cccc|c} 1 & 8/3 & 22/5 & 7 & 0.5409 \\ 3/8 & 1 & 29/8 & 14/3 & 0.2885 \\ 2/9 & 2/7 & 1 & 15/7 & 0.1088 \\ 1/7 & 1/5 & 1/2 & 1 & 0.0619 \end{array}\right), \quad \boldsymbol{A}_5^4=\left(\begin{array}{cccc|c} 1 & 2 & 10/3 & 19/4 & 0.4746 \\ 1/2 & 1 & 11/4 & 27/7 & 0.3025 \\ 1/3 & 3/8 & 1 & 5/2 & 0.1461 \\ 1/5 & 1/4 & 2/5 & 1 & 0.0768 \end{array}\right)$$

$$\boldsymbol{O}^4=\left(ICI_{kl}^4\right)_{m\times m}=\begin{pmatrix} 1.0000 & 1.0100 & 1.0079 & 1.0119 & 1.0199 \\ 1.0100 & 1.0000 & 1.0045 & 1.0267 & 1.0427 \\ 1.0079 & 1.0045 & 1.0000 & 1.0284 & 1.0278 \\ 1.0119 & 1.0267 & 1.0284 & 1.0000 & - \\ 1.0199 & 1.0427 & 1.0278 & \mathbf{1.0526} & 1.0000 \end{pmatrix}$$

Again none of the stop conditions have been met. We now have $ICI_{45}^4=\max_{D^4}\left(ICI_{kl}^4\right)$. However, the ordered pair (DM_4,DM_5) is not in D^4. The automatic feedback mechanism will just provide the suggestion to DM_5. Suppose that DM_5 rejects this suggestion, we have $D^5=D^4-(DM_4,DM_5)$, $\boldsymbol{A}_k^5=\boldsymbol{A}_k^4$, $k=1,\cdots,5$ and $\boldsymbol{O}^6=\boldsymbol{O}^5$. As $ICI_{25}^5=ICI_{52}^5=\max_{D^5}\left(ICI_{kl}^5\right)$, the feedback suggestion should be provided to DM_2 and DM_5. Suppose that only DM_2 accepts the feedback. Set $D^5=D^4-(DM_5,DM_2)$ and have

$$\boldsymbol{A}_2^6=\left(\begin{array}{cccc|c} 1 & 26/9 & 31/7 & 31/5 & 0.5450 \\ 1/3 & 1 & 23/7 & 39/8 & 0.2790 \\ 2/9 & 1/3 & 1 & 9/3 & 0.1137 \\ 1/6 & 1/5 & 3/7 & 1 & 0.0623 \end{array}\right)$$

$$\boldsymbol{O}^6=\left(ICI_{kl}^6\right)_{m\times m}=\begin{pmatrix} 1.0000 & 1.0045 & 1.0079 & 1.0119 & 1.0199 \\ 1.0045 & 1.0000 & 1.0024 & 1.0249 & 1.0240 \\ 1.0079 & 1.0024 & 1.0000 & 1.0284 & 1.0278 \\ 1.0119 & 1.0249 & 1.0284 & 1.0000 & - \\ 1.0199 & - & 1.0278 & - & 1.0000 \end{pmatrix}$$

The stop conditions are checked again and it is found that $\forall\,(DM_k,DM_l)\in D^6$, $ICI_{kl}^6<\varepsilon$. Therefore the predefined consensus level has been achieved. By using Eq. (7) and Eq.(8) we have the weight vector of decision makers $\boldsymbol{\rho}=(0.2015,0.2010,0.2006,0.1986,0.1983)^{\mathrm{T}}$. The final group PCM and priorities are

$$\boldsymbol{G}^* = \left(\begin{array}{cccc|c} 1 & 17/6 & 13/3 & 6 & 0.5430 \\ 1/3 & 1 & 4 & 4 & 0.2697 \\ 1/4 & 1/3 & 1 & 9/4 & 0.1187 \\ 1/6 & 1/4 & 4/9 & 1 & 0.0686 \end{array}\right)$$

The ranking of alternatives is $X_1 \succ X_2 \succ X_3 \succ X_4$.

6 Discussion

In this section, we discuss four features related to the proposed model and compare it with other existing group consensus models in the context of the AHP or multiplicative preference and other approaches with feedback mechanism. Compared with previous studies on consensus reaching models the proposed consensus model has the following properties:

(1) The proposed consensus reaching model is constructed to make peer to peer consensus comparisons. The reference point of consensus measuring and opinion adjusting is not needed. Therefore, we do not aggregate the individual opinions into the group opinion in each round of the consensus reaching process which will reduce the cost of the group consensus reaching process and make it easier to use, especially in the context of a large group. Furthermore, in this “Big Data” era, the preferences and opinions of decision makers will be expressed in multimedia formats. The development of cloud computing and networks reduce the cost of collecting a large group of decision makers’ opinions to make a decision. Thus in the context of group decision making with complex data and a large group, the calculation of the weight of the decision maker and the aggregated group opinion in each round of consensus reaching process would be a very formidable task. The peer to peer feature proposed in this paper is a good choice for avoiding the aggregation of the opinions within the group in every iteration. In contrast, most of the previous studies in group consensus reaching, if not all, use the aggregated group opinion as the reference point. Thus one of the contributions of the proposed model in this paper is providing a novel computable group consensus reaching framework to help a group achieve an acceptable consensus.

(2) In this consensus reaching model, the amount of opinion changing that is suggested is adaptively determined in association with the dynamic elements within the group decision environment. This adaptive model not only distinguishes different situations such as the "high" or "low" degree of consensus within a group and then the appropriate revision amount that should be chosen in order to reach a consensus, but it also takes the experts' power and the closeness with the other decision makers' opinions into consideration. In contrast to our model, Mata, et al. (2009) proposed an adaptive group consensus method in which the number of changes required for the experts in each iteration of the consensus reaching process is adapted according to the degree of consensus. Their adaptive method is focused on the number of changes that should be made in the consensus reaching process. However, our adaptive model dynamically calculates the suggested amount of opinion revision which addresses a deeper level of the decision than their model. In addition, the adaptive consensus model is the current trend in the development of consensus models and only very few results have been achieved (Herrera-Viedma, et al., 2014). Thus the proposed model is of great significance in this field.

(3) The decision makers are not forced to follow the suggestion provided by the automatic feedback mechanism in this consensus model. They are allowed to choose to revise according to the feedback suggestion or not. Therefore the consensus reaching process acts in a democratic way. This procedure may not seem like a significant contribution; however, the previous studies usually all together ignore this decision maker's right. The right of rejection is a standard to distinguish whether the group decision is made by the decision makers or made by the organizer and/or moderator. In a group decision making process, it might seem intuitive that the group preference or opinion is the collective intelligence of the individuals in this group. The moderator is not supposed to express their judgment and opinion in this process. Thus it is a violation of this principle to force the decision maker to change their opinion. Studies should clarify their point of view about the decision makers' right of rejection in the consensus

reaching process. Furthermore, if the decision makers are allowed to choose to reject or accept the feedback the proof of convergence of the consensus reaching algorithm is more complicated. The proposed model in this paper provides a solution to this dilemma.

(4) We have proven that the proposed consensus reaching process is convergent. The convergence of the proposed model, guarantees the consensus level of a group will be improved in the consensus reaching process. There are many previous studies addressing group consensus reaching models that do not provide a proof of convergence for their model. However, the convergence proof is important to demonstrate with a new group consensus reaching model. Convergence is one of the best ways to show the validity and effectiveness of a model that will be used in group decision makers.

7 Conclusions

The decision making process frequently involves multiple decision makers. One of the most widely used multi-criteria decision making methods in the group decision context is the Analytic Hierarchy Process (AHP). For determining the weights of importance of decision makers, a Markov chain weight allocating method based on the opinion transition probabilities, was presented. The selection of the incompatible decision maker is based on the individual consensus index (ICI), which dynamically changes in response to the decision makers' updated judgments. Thus the most incompatible pair of decision makers would change in different iterations of the algorithm, though the selection method is deterministic. This algorithm has been proven to be convergent and helpful to improve group consensus. In addition this algorithm has other important properties such as adaptive judgment revision, consistency preservation, and democracy, etc. Therefore, in our opinion, for a group consensus reaching problem, the proposed algorithm works very well and provides some advantages over the other classic methods in this field.

The numerical example provided shows the details of the proposed model and demonstrates its effectiveness. The authors plan to extend the proposed model to other types of preference relations. It is also an important task to develop a group decision support system based to allow the easy adoption of our model.

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Appendix A. Proof of Theorem 1

Theorem 1. In the proposed consensus reaching model, suppose all m initial PCMs $\boldsymbol{A}_1, \boldsymbol{A}_2, \cdots, \boldsymbol{A}_m$ provided by the decision makers are of acceptable consistency, then the final PCMs $\boldsymbol{A}_1^*, \boldsymbol{A}_2^*, \cdots, \boldsymbol{A}_m^*$ are of acceptable consistency.

Proof. It is not hard to prove this theorem by induction. Suppose the algorithm have T times iteration, $T \geq 0$.

(1) For $t = 0$, we get that $\boldsymbol{A}_k^0 = \boldsymbol{A}_k$ is of acceptable consistency, $k \in M$.

(2) Suppose $t \geq 0$, after the tth iteration, we will get $\boldsymbol{A}_1^{t+1}, \boldsymbol{A}_2^{t+1}, \cdots, \boldsymbol{A}_m^{t+1}$. Without loss of generality, for decision makers DM_k, $k \in M$, who do not change their PCMs, we have

$$\boldsymbol{A}_k^{t+1} = \boldsymbol{A}_k^t \tag{A.1}$$

For this case, we know that $\boldsymbol{A}_k^{t+1}$ maintain their consistency. Then for the decision makers DM_j, $j \in M$, who change their PCMs, we have

$$\boldsymbol{A}_j^{t+1} = (\boldsymbol{A}_j^t)^{\alpha} \circ (\boldsymbol{A}_l^t)^{1-\alpha} \tag{A.2}$$

where $l \in M$. Thus for this case, we get that $\boldsymbol{A}_j^{t+1}$ is the geometric combination of two PCMs with acceptable consistency. As proved by Grošelj and Zadnik Stirn (2012), we also know that $\boldsymbol{A}_j^{t+1}$ is of acceptable consistency. Summarizing both cases, we have that, $\forall k \in M$, if $\boldsymbol{A}_k^t$ is of acceptable consistency, then $\boldsymbol{A}_k^{t+1}$ is of acceptable consistency.

(3) Then we have for all $t \in [0, T]$, $\boldsymbol{A}_1^t, \boldsymbol{A}_2^t, \cdots, \boldsymbol{A}_m^t$ are all of acceptable consistency. Thus we complete the proof of **Theorem 1**. □

Appendix B. Proof of Theorem 2

Theorem 2. In the tth round of the proposed consensus reaching model, $0 \le t \le t^* - 1$, if $ICI_{kl}^t \ge \varepsilon$, $\forall k, l \in M$, then we have

$$\max_{k,l \in M} ICI_{kl}^{t+1} \le \max_{k,l \in M} ICI_{kl}^t \tag{B.1}$$

Proof. Assume that DM_p and DM_q with the maximum ICI in round t, $p, q \in M$. Thus they are selected by the feedback mechanism and may choose modify their PCMs or not. Thus in the $t+1$ round there are 3 cases of all PCMs:

(1) Both DM_p and DM_q reject to modify their judgments. We have $\boldsymbol{A}_k^{t+1} = \boldsymbol{A}_k^t$, $\forall k \in M$. Then it follows

$$\max_{k,l \in M} ICI_{kl}^{t+1} = \max_{k,l \in M} ICI_{kl}^t \tag{B.2}$$

(2) Both DM_p and DM_q revise their judgments according to the feedback suggestion, then we have

$$
\begin{aligned}
ICI_{pq}^{t+1} &= \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij(p)}^{t+1} a_{ji(q)}^{t+1} \\
&= \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=i+1}^{n}\left(a_{ij(p)}^{t+1} a_{ji(q)}^{t+1} + \frac{1}{a_{ij(p)}^{t+1} a_{ji(q)}^{t+1}} \right) + \frac{1}{n} \\
&= \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=i+1}^{n}\left(\left(a_{ij(p)}^{t}\right)^{\alpha_p^t} \left(a_{ij(q)}^{t}\right)^{1-\alpha_p^t} \left(a_{ji(q)}^{t}\right)^{\alpha_q^t} \left(a_{ji(p)}^{t}\right)^{1-\alpha_q^t} + \frac{1}{a_{ij(p)}^{t+1} a_{ji(q)}^{t+1}} \right) + \frac{1}{n} \qquad \text{(B.3)} \\
&= \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=i+1}^{n}\left(\left(a_{ij(p)}^{t} a_{ji(q)}^{t}\right)^{\alpha_p^t+\alpha_q^t-1} + \frac{1}{\left(a_{ij(p)}^{t} a_{ji(q)}^{t}\right)^{\alpha_p^t+\alpha_q^t-1}} \right) + \frac{1}{n} \\
&\le \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=i+1}^{n}\left(a_{ij(p)}^{t} a_{ji(q)}^{t} + \frac{1}{a_{ij(p)}^{t} a_{ji(q)}^{t}} \right) + \frac{1}{n}
\end{aligned}
$$

where the inequality in Eq. (B.3) is from

$$x^{\theta} + \frac{1}{x^{\theta}} \le x + \frac{1}{x}, \forall x > 0, 0 < \theta < 1 \qquad \text{(B.4)}$$

Since $ICI_{pq}^{t} > 1$, then at least we have one pair (i,j) such that $a_{ij(p)}^{t} a_{ji(q)}^{t} \neq 1$. Thus the inequality strictly holds and we have

$$ICI_{pq}^{t+1} < \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=i+1}^{n}\left(a_{ij(p)}^{t} a_{ji(q)}^{t} + \frac{1}{a_{ij(p)}^{t} a_{ji(q)}^{t}} \right) + \frac{1}{n} = ICI_{pq}^{t} \qquad \text{(B.5)}$$

Then $\forall k \in M, k \neq q$, we have

$$
\begin{aligned}
ICI_{pk}^{t+1} &= \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij(p)}^{t+1} a_{ji(k)}^{t+1} \\
&= \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} \left(a_{ij(p)}^{t}\right)^{\alpha_p^t} \left(a_{ij(q)}^{t}\right)^{1-\alpha_p^t} a_{ji(k)}^{t} \\
&= \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} \left(a_{ij(p)}^{t} a_{ji(k)}^{t}\right)^{\alpha_p^t} \left(a_{ij(q)}^{t} a_{ji(k)}^{t}\right)^{1-\alpha_p^t} \\
&\le \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left[\alpha_p^t a_{ij(p)}^{t} a_{ji(k)}^{t} + (1-\alpha_p^t) a_{ij(q)}^{t} a_{ji(k)}^{t} \right] \qquad \text{(B.6)} \\
&= \alpha_p^t ICI_{pk}^{t} + (1-\alpha_p^t) ICI_{qk}^{t} \\
&\le \max\left\{ ICI_{pk}^{t}, ICI_{qk}^{t} \right\} \\
&< ICI_{pq}^{t}
\end{aligned}
$$

where the first inequality in Eq.(B.6) is from the inequality of the weighted arithmetic and geometric mean. Similar to Eq. (B.6), $\forall k \in M, k \neq p$, we have

$$ICI_{qk}^{t+1} < ICI_{pq}^{t} \qquad \text{(B.7)}$$

Then we have

$$\begin{aligned}\max_{k,l\in M} ICI_{kl}^{t+1} &= \max\left(\max_{k,l\in M,k,l\neq p,q} ICI_{kl}^{t+1}, \max_{k\in M,k\neq q} ICI_{pk}^{t+1}, \max_{k\in M,k\neq q} ICI_{qk}^{t+1}, ICI_{pq}^{t+1}\right)\\ &< \max\left(\max_{k,l\in M,k,l\neq p} ICI_{kl}^{t}, ICI_{pq}^{t}\right)\\ &= ICI_{pq}^{t}\\ &= \max_{k,l\in M} ICI_{kl}^{t}\end{aligned} \tag{B.8}$$

(3) One of the two selected decision makers modified his\her PCM and the other did not. Without loss of generality, we assume that DM_p accepted the feedback suggestion in round t. Thus from Eq. (B.6) and Eq.(B.7), we have

$$\begin{aligned}\max_{k,l\in M} ICI_{kl}^{t+1} &= \max\left(\max_{k,l\in M,k,l\neq p} ICI_{kl}^{t+1}, \max_{k\in M} ICI_{pk}^{t+1}\right)\\ &< \max\left(\max_{k,l\in M,k,l\neq p} ICI_{kl}^{t}, ICI_{pq}^{t}\right)\\ &= ICI_{pq}^{t}\\ &= \max_{k,l\in M} ICI_{kl}^{t}\end{aligned} \tag{B.9}$$

Summarizing all 3 cases, we can complete the proof of **Theorem 2**. □

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