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Mathematics and Computers in Simulation 80 (2009) 378–386
MATHEMATICS AND COMPUTERS IN SIMULATION
www.elsevier.com/locate/matcom

Option pricing under the Merton model of the short rate

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Received 20 April 2009; received in revised form 11 July 2009; accepted 25 July 2009
Available online 3 August 2009

Abstract

Previous option pricing research typically assumes that the risk-free rate or the short rate is constant during the life of the option. In this study, we incorporate the stochastic nature of the short rate in our option valuation model and derive explicit formulas for European call and put options on a stock when the short rate follows the Merton model. Using our option model as a benchmark, our numerical analysis indicates that, in general, the Black–Scholes model overvalues out-of-the-money calls, moderately overvalues at-the-money calls, and slightly overvalues in-the-money calls. Our analysis is directly extensible to American calls on non-dividend-paying stocks and to European puts by virtue of put-call parity.

Keywords: Option pricing; Merton short rate model; Black–Scholes option model; Pricing biases

1. Introduction

Since Black and Scholes [2] made an important breakthrough by deriving an exact no-arbitrage pricing formula for European options, many academics have worked on option pricing research and come up with alternative formulas to the original Black–Scholes (B–S) pricing formula. Among these academics are Cox and Ross [5], Merton [21], Roll [22], Cox et al. [6], Geske [8], Lee et al. [15], Whaley [26], Jarrow and Rudd [11], Rubinstein [23], Hull and White [10], Johnson and Shanno [12], Johnson and Stulz [13], Scott [24], Wiggins [27], Duan [7], and Heston and Nandi [9], each making different assumptions about the various factors that affect the price of an option. However, all the above option pricing studies assume that the risk-free rate or the short rate is constant during the life of the option.

The assumption of a constant short rate $r(t)$ is clearly at odds with reality because, as a matter of fact, $r(t)$ is evolving randomly over time. Fig. 1 shows the 3-month U.S. Treasury bill rate (often used as a proxy for the short rate) from 1999 to 2008, fluctuating randomly between 0.00 and 0.0625. Hence, in this study, we incorporate its stochastic nature into our option valuation model. Specifically, we use the following stochastic process, first proposed by Merton [19,20], to depict its dynamics and derive explicit pricing formulas for European call and put on a stock.

$$dr(t) = \alpha dt + \sigma dZ_r \tag{1}$$

where α is the drift in the short rate, σ is the volatility of the short rate, and Z_r is a standard Wiener process. Henceforth, the stochastic process in Eq. (1) will be referred to as the Merton model of the short rate.

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doi:10.1016/j.matcom.2009.07.006

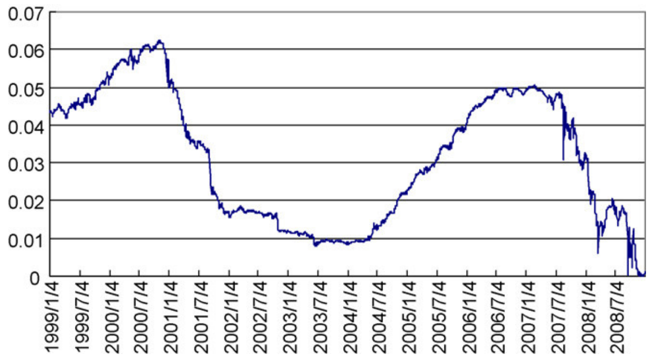

Fig. 1. Three-month U.S. Treasury bill rate from 1999 to 2008.

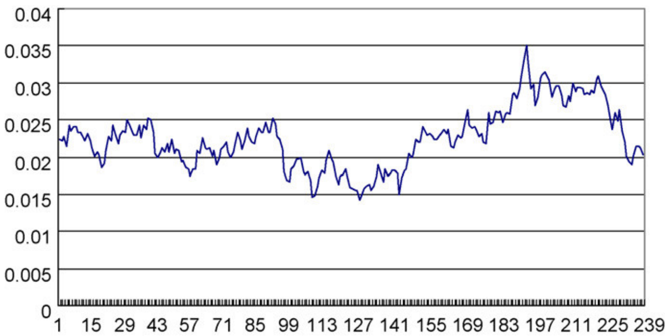

Fig. 2. A sample path of the short rate under the Merton model.

In theory, the Merton model implies a positive probability of negative interest rates. But negative rates are less likely to occur if α is positive. In addition, most exchange-traded stock options have a maturity of less than 9 months. Hence, using an initial positive value for $r(t)$ and appropriate values for α and σ, we can be assured that its expected first-passage time to the origin is longer than 9 months. Fig. 2 shows a sample path of $r(t)$ over a 1-year duration based on $r(t)$ = short rate at initial time $t=0.02$, $\Delta t=0.004$, $\bar{\alpha} = 0.005$, and $\bar{\sigma} = 0.02$, where $\bar{\alpha}$ and $\bar{\sigma}$ are estimated by Chan et al. [4] using the generalized method of moments.

There is considerable empirical evidence[1] of systematic biases in the B–S model. For example, Merton [21] points out that the market prices for deep-in-the-money, deep-out-of-the-money, and shorter-maturity calls tend to sell for more than the B–S prices, whereas the market prices for marginally-in-the-money and longer-maturity calls tend to sell for less than the B–S prices. MacBeth and Merville [16] report that the B–S prices are, on average, greater (less) than the market prices for out-of-the-money (in-the-money) calls. In Section 3, we will employ our option valuation model as a benchmark and investigate the pricing biases in the B–S model.

The rest of the paper proceeds as follows: In Section 2, we derive the formula for the price of a riskless zero-coupon bond paying \$1 at maturity based on Eq. (1) and then derive the pricing formulas for European call and put on a stock. In Section 3, we examine how the various parameters of our model affect option prices and investigate the pricing biases in the B–S model when we assume that the correct model involves a stochastic short rate following Eq. (1). Section 4 concludes this research.

2. Derivation of option pricing formulas under the Merton model

To proceed, we first derive the pricing formula for a riskless zero-coupon bond under the Merton model in Subsection 2.1 and then, using the bond pricing formula, derive explicit formulas for European call and put options in Subsection 2.2.

[1] See Black [1], Merton [21], MacBeth and Merville [16,17], and Lauterbach and Schultz [14].

2.1. Pricing formula for a zero-coupon bond

Let $P_b = P_b(r,t,T)$ be the price at time t of a riskless zero-coupon bond paying \$1 at time T, where we set $\tau = T - t$. That is, $P_b(r,T,T) = 1$. Using the Merton model of the short rate in Eq. (1) and applying Ito's lemma, we have

$$dP_b = \left[\frac{\partial P_b}{\partial t} + \alpha\frac{\partial P_b}{\partial r} + \frac{1}{2}\sigma^2\frac{\partial^2 P_b}{\partial r^2}\right]dt + \sigma\frac{\partial P_b}{\partial r}dZ_r \tag{2}$$

Letting $\mu_b = 1/P_b[\partial P_b/\partial t + \alpha(\partial P_b/\partial r) + 1/2\sigma^2(\partial^2 P_b/\partial r^2)]$, $\sigma_b = -1/P_b[\sigma(\partial P_b/\partial r)] = -\sigma/P_b[\partial P_b/\partial r]$, and assuming the local expectations hypothesis holds for the term structure of interest rates (i.e., $\mu_b = r$), we obtain

$$\frac{dP_b}{P_b} = rdt + \sigma_b dZ_r \tag{3}$$

$$\frac{1}{2}\sigma^2\frac{\partial^2 P_b}{\partial r^2} + \alpha\frac{\partial P_b}{\partial r} + \frac{\partial P_b}{\partial t} - rP_b = 0 \tag{4}$$

To solve Eq. (4) for $P_b(r,t,T)$, let $\tau = T - t$ and $P_b(r,t,T) = \exp[A(\tau) - rB(\tau)]$. Then we have $\partial P_b/\partial r = -B(\tau)P_b$, $\partial^2 P_b/\partial r^2 = B^2(\tau)P_b$, and $\partial P_b/\partial t = P_b[(-\partial A(\tau)/\partial t) + r(\partial B(\tau)/\partial t)]$. Substituting them into Eq. (4) and simplifying, Eq. (4) becomes

$$P_b\left\{\left[\frac{1}{2}\sigma^2 B^2(\tau) - \alpha B(\tau) - \frac{\partial A(\tau)}{\partial t}\right] + r\left[\frac{\partial B(\tau)}{\partial t} - 1\right]\right\} = 0 \tag{5}$$

Eq. (5) implies that $1/2\sigma^2 B^2(\tau) - \alpha B(\tau) - (\partial A(\tau)/\partial t) = 0$ and $\partial B(\tau)/\partial t - 1 = 0$. Solving for $A(\tau)$ and $B(\tau)$, we have $A(\tau) = -\alpha\tau^2/2 + \sigma^2\tau^3/6$ and $B(\tau) = \tau$. Hence, we obtain a formula for the price at time t of a riskless zero-coupon bond paying \$1 at maturity T.

$$P_b(r, t, T) = \exp\left\{-r\tau - \frac{\alpha\tau^2}{2} + \frac{\sigma^2\tau^3}{6}\right\} \tag{6}$$

2.2. Derivation of explicit formulas for European call and put options

In this study, we assume that there are no transaction costs, margin requirements, and taxes; all securities are divisible; security trading is continuous and borrowing and short-selling are permitted without restriction; there are no dividend payouts over the life of the option; and all investors can borrow or lend at the same short rate. Further, we assume that the stock price follows the following geometric Wiener process:

$$\frac{dP_s}{P_s} = \mu_s dt + \sigma_s dZ_s \tag{7}$$

where μ_s and σ_s are constant, and Z_s is a standard Wiener process. Further, the correlation between dZ_r and dZ_s is given by ρ.

Let X be the exercise price of the call option and $c = c(P_s,P_b,\tau;X)$ be the call price, which is a function of the stock price P_s, the riskless zero-coupon bond price P_b, and the time to maturity τ. By Ito's lemma, the change in the call price over an infinitesimal time dt satisfies the following stochastic differential equation:

$$dc = \frac{\partial c}{\partial P_s}dP_s + \frac{\partial c}{\partial P_b}dP_b + \frac{\partial c}{\partial \tau}d\tau + \frac{\partial^2 c}{\partial P_s\partial P_b}(dP_s dP_b) + \frac{1}{2}\left[\frac{\partial^2 c}{\partial P_s^2}(dP_s)^2 + \frac{\partial^2 c}{\partial P_b^2}(dP_b)^2\right] \tag{8}$$

Substituting $dP_s dP_b = \rho\sigma_s\sigma_b P_s P_b dt$, $(dP_s)^2 = \sigma_s^2 P_s^2 dt$, $(dP_b)^2 = \sigma_b^2 P_b^2 dt$, and $d\tau = -dt$ into Eq. (8), we have

$$dc = \frac{\partial c}{\partial P_s}dP_s + \frac{\partial c}{\partial P_b}dP_b + \left[\frac{1}{2}\frac{\partial^2 c}{\partial P_s^2}\sigma_s^2 P_s^2 + \frac{1}{2}\frac{\partial^2 c}{\partial P_b^2}\sigma_b^2 P_b^2 + \frac{\partial^2 c}{\partial P_s\partial P_b}\rho\sigma_s\sigma_b P_s P_b - \frac{\partial c}{\partial t}\right]dt \tag{9}$$

Now we form a hedge portfolio consisting of the stock, the riskless bond, and the call. Let Q_s be the number of shares of the stock, Q_b be the number of the bond, and Q_c be the number of the call. The hedge is formed such that the value (say, H) of the hedge portfolio is zero.[2] That is, $H \equiv Q_sP_s + Q_bP_b + Q_cc = 0$. Hence, we have

$$dH = Q_s dP_s + Q_b dP_b + Q_c dc = 0 \tag{10}$$

Substituting Eq. (9) into Eq. (10) and grouping, Eq. (10) becomes

$$\begin{aligned} dH = {} & Q_c \left[\frac{1}{2}\frac{\partial^2 c}{\partial P_s^2}\sigma_s^2 P_s^2 + \frac{1}{2}\frac{\partial^2 c}{\partial P_b^2}\sigma_b^2 P_b^2 + \frac{\partial^2 c}{\partial P_s \partial P_b}\rho\sigma_s\sigma_b P_s P_b - \frac{\partial c}{\partial \tau} \right] dt \\ & + \left[Q_c \frac{\partial c}{\partial P_s} + Q_s \right] dP_s + \left[Q_c \frac{\partial c}{\partial P_b} + Q_b \right] dP_b = 0 \end{aligned} \tag{11}$$

Eq. (11) implies that $Q_c(\partial c/\partial P_s) + Q_s = 0$, $Q_c(\partial c/\partial P_b) + Q_b = 0$, and

$$\frac{1}{2}\frac{\partial^2 c}{\partial P_s^2}\sigma_s^2 P_s^2 + \frac{1}{2}\frac{\partial^2 c}{\partial P_b^2}\sigma_b^2 P_b^2 + \frac{\partial^2 c}{\partial P_s \partial P_b}\rho\sigma_s\sigma_b P_s P_b - \frac{\partial c}{\partial \tau} = 0 \tag{12}$$

Hence, the price of a European call must satisfy Eq. (12) subject to the following two boundary conditions: $c(0,P_b,\tau;X) = 0$ and $c(P_s,1,\tau;X) = \max(0,P_s - X)$.

To solve for c in Eq. (12), we have to transform Eq. (12) to a standard one-dimensional heat equation[3] of the form $\partial u(x,t)/\partial t = k(\partial^2 u(x,t)/\partial x^2)$, where k is some constant. We make use of the linear homogeneity[4] of c in P_s and XP_b so that we can perform such transformation for Eq. (12). Accordingly, we set $\Theta \equiv \Theta(P_s,P_b,t) \equiv P_s/XP_b$. By Ito's lemma and Eqs. (3) and (7), the total differential of Θ is given by

$$\begin{aligned} d\Theta = {} & \left[\frac{\partial\Theta}{\partial P_s}\mu_s P_s + \frac{\partial\Theta}{\partial P_b} r P_b + \frac{\partial\Theta}{\partial t} + \frac{1}{2}\frac{\partial^2\Theta}{\partial P_s^2}\sigma_s^2 P_s^2 + \frac{1}{2}\frac{\partial^2\Theta}{\partial P_b^2}\sigma_b^2 P_s^2 + \frac{\partial^2\Theta}{\partial P_s \partial P_b}\rho\sigma_s\sigma_b P_s P_b \right] dt \\ & + \frac{\partial\Theta}{\partial P_s}\sigma_s P_s dZ_s + \frac{\partial\Theta}{\partial P_b}\sigma_b P_b dZ_r \end{aligned} \tag{13}$$

Substituting $\partial\Theta/\partial P_s = 1/XP_b$, $\partial\Theta/\partial P_b = -P_s/XP_b^2$, $\partial\Theta/\partial t = 0$, $\partial^2\Theta/\partial P_s^2 = 0$, $\partial^2\Theta/\partial P_b^2 = P_s/XP_b^3$, $\partial^2\Theta/\partial P_s\partial P_b = -1/XP_b^2$ into Eq. (13) and simplifying, Eq. (13) becomes

$$\frac{d\Theta}{\Theta} = \mu_\Theta dt + \sigma_\Theta dZ_\Theta \tag{14}$$

where $\mu_\Theta = \mu_s - r + (\sigma_b^2/2) - \rho\sigma_s\sigma_b$ and $\sigma_\Theta^2 = \sigma_s^2 + \sigma_b^2 - 2\rho\sigma_s\sigma_b$.

To solve Eq. (12) for $c(P_s,P_b,\tau;X)$ subject to the two boundary conditions, we use a new variable C such that $C \equiv C(\Theta,\tau;X) \equiv c(P_s,P_b,\tau;X)/XP_b$. That is, C is the call price[5] expressed in the same units as Θ. That is, $c(P_s,P_b,\tau;X) = XP_bC(\Theta,\tau;X)$. Then $\partial^2 c/\partial P_s^2 = (1/XP_b)\partial^2 C/\partial P_s^2$, $\partial^2 c/\partial P_b^2 = (P_s^2/XP_b^3)\partial^2 C/\partial P_s^2$, $\partial^2 c/\partial P_s\partial P_b = (-P_s/XP_b^2)\partial^2 C/\partial P_s^2$, and $\partial c/\partial\tau = XP_b(\partial C/\partial\tau)$. Substituting them into Eq. (12) and simplifying, Eq. (12) becomes

$$\frac{1}{2}\left(\frac{P_s}{XP_b}\right)^2 [\sigma_s^2 + \sigma_b^2 - 2\rho\sigma_s\sigma_b]\frac{\partial^2 C}{\partial P_s^2} - \frac{\partial C}{\partial\tau} = 0 \tag{15}$$

Since $\Theta^2 = (P_s/XP_b)^2$ and $\sigma_\Theta^2 = \sigma_s^2 + \sigma_b^2 - 2\rho\sigma_s\sigma_b$, Eq. (15) becomes

$$\frac{1}{2}\sigma_\Theta^2\Theta^2\frac{\partial^2 C}{\partial\Theta^2} - \frac{\partial C}{\partial\tau} = 0 \tag{16}$$

That is, $C = C(\Theta,\tau;X)$ must satisfy Eq. (16) subject to the following two boundary conditions: $C(0,\tau;X) = 0$ and $C(\Theta,0;X) = \max(0,\Theta - 1)$.

[2] Operationally, this hedge can be formed by using the proceeds from borrowing at the short rate and short-sales to pay for the long positions.

[3] See, for example, Chapter 6 of Brown and Churchill [3].

[4] See Margrabe [18] for more on linear homogeneity.

[5] This is feasible because $c(P_s,P_b,\tau;X)$ is homogeneous of degree one in P_s and XP_b.

Now defining a new variable $\Sigma \equiv \int_t^T \sigma_\Theta^2(\varphi)d\varphi$ and, accordingly, defining $\Phi(\Theta,\Sigma)\equiv C(\Theta,\tau)$, we have $\partial^2 C/\partial\Theta^2 = \partial^2\Phi/\partial\Theta^2$ and $\partial C/\partial\tau = (\partial\Phi/\partial\Sigma)(\partial\Sigma/\partial\tau) = (\partial\Phi/\partial\Sigma)\sigma_\Theta^2$. Substituting them into Eq. (16) and simplifying, we have

$$\frac{1}{2}\Theta^2\frac{\partial^2\Phi}{\partial\Theta^2} - \frac{\partial\Phi}{\partial\Sigma} = 0 \tag{17}$$

To solve Eq. (17) subject to $\Phi(0,\Sigma)=0$ and $\Phi(\Theta,0)=\max(0,\Theta-1)$, we transform Eq. (17) by making the change of variables $\theta\equiv\log\Theta+(\Sigma/2)$ and $u(\theta,\Sigma)\equiv\Phi(\Theta,\Sigma)/\Theta$. We have $\Phi(\Theta,\Sigma) = u(\log\Theta + 1/2\int_t^T \sigma_\Theta^2(\varphi)d\varphi, \int_t^T \sigma_\Theta^2(\varphi)d\varphi)\Theta$, $\partial^2\Phi/\partial\Theta^2 = 1/\Theta(\partial u/\partial\theta + \partial^2 u/\partial\theta^2)$, and $\partial\Phi/\partial\Sigma = \Theta(1/2\partial u/\partial\theta + \partial u/\partial\Sigma)$. Substituting them into Eq. (17) and simplifying, Eq. (17) becomes $\partial u/\partial\Sigma = 1/2(\partial^2 u/\partial\theta^2)$. For the first boundary condition, note that $u(\Theta,\Sigma)=\Phi(\Theta,\Sigma)/\Theta=(1/\Theta)C(\Theta,\tau)=1/\Theta(c/XP_b)=(XP_b/P_s)c/XP_b=c/P_s$. Thus the first boundary condition is $|u(\theta,\Sigma)|=(c/P_s)\le 1$ and the second boundary condition is $1-(1/\Theta)=1-\exp[-\log\Theta]=1-\exp[-\theta]$, because $\theta=\log\Theta$ when $\Sigma=0$. Hence, $u(\theta,\Sigma=0)=(\Theta-1)/\Theta=1-(1/\Theta)=1-\exp[-\theta]$ if $1\ge\exp[-\theta]$ and $\Phi(\theta,\Sigma=0)=0$ if $1<\exp[-\theta]$. In sum, our boundary value problem consists of a boundedness condition $|u(\theta,\Sigma)|\le 1$ and the following two conditions:

$$\frac{\partial u(\theta,\Sigma)}{\partial\Sigma} = \frac{1}{2}\frac{\partial^2(\theta,\Sigma)}{\partial\theta^2} \tag{18}$$

$$\begin{aligned} u(\theta,0) &= 1-\exp[-\theta] \quad \text{if } 1\ge\exp[-\theta] \\ &= 0 \quad\quad\quad\quad\quad\;\; \text{if } 1<\exp[-\theta] \end{aligned} \tag{19}$$

Eq. (18) is in standard one-dimensional heat equation and thus can be solved. Let $u(\theta,\Sigma)=f(\theta)g(\Sigma)$, where $f(\theta)$ is some function of θ and $g(\Sigma)$ is some function of Σ. Substituting $u(\theta,\Sigma)=f(\theta)g(\Sigma)$ into Eq. (18) and simplifying, we obtain

$$\frac{2}{g(\Sigma)}\frac{\partial g(\Sigma)}{\partial\Sigma} = \frac{1}{f(\theta)}\frac{\partial^2 f(\theta)}{\partial\theta^2} \tag{20}$$

That is, the left side of Eq. (20) depends only on Σ and the right side depends only on θ. Thus, we can set both sides of Eq. (20) equal to a constant[6] $-k^2$, where $k>0$. By simplifying, we obtain the following two ordinary differential equations:

$$\frac{\partial g(\Sigma)}{\partial\Sigma} + \frac{1}{2}k^2 g(\Sigma) = 0 \tag{21}$$

$$\frac{\partial^2 f(\theta)}{\partial\theta^2} + k^2 f(\theta) = 0 \tag{22}$$

Solving Eqs. (21) and (22), we have $g(\Sigma)=\exp(-1/2k^2\Sigma)$ and $f(\theta)=A(\theta)\cos(k\theta)+B(\theta)\sin(k\theta)$. The linear combination of functions $u(\theta,\Sigma)=f(\theta)g(\Sigma)$ becomes

$$u(\theta,\Sigma) = \int_0^\infty [A(k)\cos(k\theta) + B(k)\sin(k\theta)]\exp\left(-\frac{1}{2}k^2\Sigma\right)dk \tag{23}$$

[6] We set both sides of Eq. (20) equal to $-k^2$, where $k>0$, in order that the two ordinary differential equations in Eqs. (21) and (22) have continuous eigenvalues k^2.

If $A(k) = 1/\pi \int_{-\infty}^{\infty} f(\omega)\cos(k\omega)\,d\omega$ and $B(k) = 1/\pi \int_{-\infty}^{\infty} f(\omega)\sin(k\omega)\,d\omega$, Eq. (23) is valid according to the Fourier integral theorem. Substituting them into Eq. (23), we obtain[7]

$$\begin{aligned} u(\theta, \Sigma) &= \frac{1}{\pi}\int_0^{\infty}\left\{\int_{-\infty}^{\infty}[\cos(k\omega)\cos(k\theta)+\sin(k\omega)\sin(k\theta)]\,f(\omega)d\omega\right\}\exp\left(-\frac{1}{2}k^2\Sigma\right)dk \\ &= \frac{1}{\sqrt{2\Sigma\pi}}\int_{-\infty}^{\infty} f(\omega)\exp\left(-\frac{(\omega-\theta)^2}{2\Sigma}\right)d\omega \end{aligned} \tag{24}$$

Letting $q = (\omega-\theta)/\sqrt{2\Sigma}$, we have $\omega = \theta + q\sqrt{2\Sigma}$ and $d\omega = \sqrt{2\Sigma}dq$. Substituting $q = (\omega-\theta)/\sqrt{2\Sigma}$ and Eq. (20) into Eq. (24), we obtain

$$\begin{aligned} u(\theta, \Sigma) &= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} f(\theta+q\sqrt{2\Sigma})\exp(-q^2)\,dq = \frac{1}{\sqrt{\pi}}\int_{-\theta/\sqrt{2\Sigma}}^{\infty}[1-\exp(-\theta-q\sqrt{2\Sigma})]\exp(-q^2)\,dq \\ &= \frac{1}{\sqrt{\pi}}\int_{-\theta/\sqrt{2\Sigma}}^{\infty}\exp(-q^2)dq - \frac{1}{\sqrt{\pi}}\int_{-\theta/\sqrt{2\Sigma}}^{\infty}\exp(-\theta-q\sqrt{2\Sigma}-q^2)\,dq \end{aligned} \tag{25}$$

To solve Eq. (25), we make another change of variables by letting $a = \sqrt{2}q$. Then the first term of Eq. (25) becomes

$$\frac{1}{\sqrt{\pi}}\int_{-\theta/\sqrt{2\Sigma}}^{\infty}\exp(-q^2)\,dq = \frac{1}{\sqrt{2\pi}}\int_{-\theta/\sqrt{\Sigma}}^{\infty}\exp\left(-\frac{a^2}{2}\right)da = N\left(\frac{\theta}{\sqrt{\Sigma}}\right) = N(a_1) \tag{26}$$

where $a_1 = \theta/\sqrt{\Sigma}$ and $N(\cdot)$ is the cumulative probability distribution function for a standardized normal random variable.

For the second term of Eq. (25), $-\theta - q\sqrt{2\Sigma} - q^2 = -\log\Theta - (\Sigma/2) - a\sqrt{\Sigma} - (a^2/2) = \log 1/\Theta - (1/2)[a^2 + 2a\sqrt{\Sigma} + (\sqrt{\Sigma})^2] = \log 1/\Theta - 1/2(a+\sqrt{\Sigma})^2$. So the integrand of the second term is $\exp(-\theta - q\sqrt{2\Sigma} - q^2) = \exp[\log 1/\Theta - (1/2)(a+\sqrt{\Sigma})^2] = 1/\Theta\exp[-1/2(a+\sqrt{\Sigma})^2]$. Substituting it into the second term of Eq. (25) and simplifying, we have

$$\begin{aligned} \frac{1}{\sqrt{\pi}}\int_{-\theta/\sqrt{2\Sigma}}^{\infty}\exp(-\theta-q\sqrt{2\Sigma}-q^2)dq &= \frac{1}{\Theta\sqrt{2\pi}}\int_{-\infty}^{a_1}\exp\left[-\frac{1}{2}(a+\sqrt{\Sigma})^2\right]da \\ &= \frac{1}{\Theta\sqrt{2\pi}}\int_{-\infty}^{a_2}\exp\left[-\frac{b^2}{2}\right]db = \frac{1}{\Theta}N(a_2) \end{aligned} \tag{27}$$

where $b = a + \sqrt{\Sigma}$ and $a_2 = a_1 - \sqrt{\Sigma}$. Combining Eqs. (26) and (27), we have

$$u(\theta, \Sigma) = N(a_1) - \frac{1}{\Theta}N(a_2) \tag{28}$$

$$\text{or} \quad \Phi(\Theta, \Sigma) = \Theta u(\theta, \Sigma) = \Theta N(a_1) - N(a_2) \tag{29}$$

Note that $\Theta \equiv P_s/XP_b$ and $c(P_s,P_b,\tau{:}X) \equiv XP_bC(\Theta,\tau{:}X) \equiv XP_s\,\Phi(\Theta,\Sigma)$. By the linear homogeneity property of $c \equiv call_O$, the price of a European call is

$$c = XP_s[\Theta N(a_1) - N(a_2)] = P_sN(a_1) - XP_bN(a_2) \tag{30}$$

where $a_1 = \theta/\sqrt{\Sigma} = [\log\Theta + (\Sigma/2)]/\sqrt{\Sigma}$ and $a_2 = a_1 - \sqrt{\Sigma} = [\log\Theta - (\Sigma/2)]/\sqrt{\Sigma}$. Further, Σ is given by

$$\Sigma \equiv \int_t^T \sigma_\Theta^2(\varphi)d\varphi = \int_t^T[\sigma_s^2 + \sigma_b^2 - 2\rho\sigma_s\sigma_b]d\varphi = \sigma_s^2\tau + \sigma^2\tau^3 - 2\rho\sigma_s\sigma\tau^2.$$

By put-call parity, the price $p \equiv p(P_s,P_b,\tau;X) \equiv put_O$ of a European put is

$$p = c - P_s + XP_b = XP_bN(-a_2) - P_sN(-a_1) \tag{31}$$

[7] Detailed derivation of Eq. (24) can be obtained from the first author upon request.

Table 1
Prices for out-of-the-money calls when the short rate at initial time t is 0.06.

τ	ρ	α	σ	$call_O$	$call_{BS}$	Δ	$\Delta\%$
0.25	0.2	0.002	0.02	0.0345	0.0351	0.0006	1.7
	0.2	0.002	0.08	0.0330	0.0351	0.0021	6.4
	0.2	0.008	0.02	0.0346	0.0351	0.0005	1.4
	0.2	0.008	0.08	0.0331	0.0351	0.0020	6.0
	0.8	0.002	0.02	0.0326	0.0351	0.0025	7.7
	0.8	0.002	0.08	0.0257	0.0351	0.0094	36.6
	0.8	0.008	0.02	0.0327	0.0351	0.0024	7.3
	0.8	0.008	0.08	0.0258	0.0351	0.0093	36.0
0.75	0.2	0.002	0.02	0.4044	0.4128	0.0084	2.1
	0.2	0.002	0.08	0.3906	0.4128	0.0222	5.7
	0.2	0.008	0.02	0.4090	0.4128	0.0038	0.9
	0.2	0.008	0.08	0.3951	0.4128	0.0177	4.5
	0.8	0.002	0.02	0.3705	0.4128	0.0423	11.4
	0.8	0.002	0.08	0.2550	0.4128	0.1578	61.9
	0.8	0.008	0.02	0.3749	0.4128	0.0379	10.1
	0.8	0.008	0.08	0.2587	0.4128	0.1541	59.6

Note that $P_s = 15$, $X = 20$, $\sigma_s = 0.3$, $call_O$ = call price for our model, $call_{BS}$ = call price for B–S model, $\Delta = call_{BS} - call_O$, $\Delta\% = 100\Delta/call_O$, and τ = time to maturity (in year).

If the short rate is constant (i.e., α and σ in Eq. (1) are both 0), then the bond price in Eq. (6) is $P_b(r,t,T) = \exp(-r\tau)$ and $\Sigma = \int_t^T \sigma_s^2 d\varphi = \sigma_s^2\tau$. Substituting them into Eqs. (30) and (31), we obtain the B–S formulas for European call and put options.

$$c_{BS} \equiv call_{BS} = P_s N(d_1) - X\ \exp(-r\tau)N(d_2) \tag{32}$$

$$p_{BS} \equiv put_{BS} = X\ \exp(-r\tau)N(-d_2) - P_s N(-d_1) \tag{33}$$

where $d_1 = (\log\ P_s - \log\ X + (r + \sigma_s^2/2)\tau)/\sigma_s\sqrt{\tau}$ and $d_2 = d_1 - \sigma_s\sqrt{\tau}$.

3. Effect of the parameters on option price and pricing biases in the B–S model

For a non-dividend-paying stock, the prices for its European puts and calls are connected by put-call parity.[8] As such, for space reason, we focus only on analyzing calls. We point out how the various parameters affect call price and then investigate the pricing biases in the B–S model. We focus on out-of-the-money, at-the-money, and in-the-money calls when the short rate[9] at initial time is 0.06. To examine call prices under different but reasonable parameterizations of the short rate, we set $\alpha = 0.002$ and 0.008, $\sigma = 0.02$ and 0.08, $\rho = 0.2$ and[10] 0.8, and $\tau = 0.25$ and 0.75 year.

From Eqs. (6) and (30), we note that each of the parameters exerts different degree of effect on the call price c. Specifically, c increases as ρ or σ decreases and decreases as ρ or σ increases; on the other hand, c increases as $r(t)$, α, or τ increases and decreases as $r(t)$, α, or τ decreases. The reason that c increases as ρ decreases is as follows: given that $\Sigma = \sigma_s^2\tau + \sigma^2\tau^3 - 2\rho\sigma_s\sigma\tau^2$ and $\sigma^2\tau^3 < 2\rho\sigma_s\sigma\tau^2$, we have that Σ increases as ρ decreases and Σ decreases as ρ increases. Since c increases as Σ increases, call price increases as ρ decreases. The reason that c increases as τ increases is as follows: given that $\Sigma = \sigma_s^2\tau + \sigma^2\tau^3 - 2\rho\sigma_s\sigma\tau^2$ and $-r\tau_1 - (\alpha\tau_1^2/2) + (\sigma^2\tau_1^3/6) > -r\tau_2 - (\alpha\tau_2^2/2) + (\sigma^2\tau_2^3/6)$ for $\tau_1 < \tau_2$, we have that Σ increases as τ increases and P_b decreases as τ increases. Since c increases as Σ increases and/or P_b decreases, call price increases as τ increases. Similar reasoning can be made for the other three parameters.

Table 1 shows the prices based on Eq. (30) and the B–S prices based on Eq. (32) for out-of-the-money ($P_s = 15 < X = 20$) calls. Based on Eq. (30), the B–S model overvalues out-of-the-money calls. For example,

[8] This parity relationship between call and put prices was first pointed out by Stoll [25].
[9] Similar results can be obtained for call options when $r(t) = 0.02$ or 0.10.
[10] Note that stock and bond prices tend to increase as interest rates decrease and to decrease as interest rates increase. Hence, stock and bond prices are generally positively correlated, which means $\rho > 0$.

Table 2
Prices for at-the-money calls when the short rate at initial time t is 0.06.

τ	ρ	α	σ	$call_O$	$call_{BS}$	Δ	$\Delta\%$
0.25	0.2	0.002	0.02	1.3385	1.3416	0.0031	0.2
	0.2	0.002	0.08	1.3289	1.3416	0.0127	1.0
	0.2	0.008	0.02	1.3403	1.3416	0.0013	0.1
	0.2	0.008	0.08	1.3308	1.3416	0.0108	0.8
	0.8	0.002	0.02	1.3266	1.3416	0.0150	1.1
	0.8	0.002	0.08	1.2802	1.3416	0.0614	4.8
	0.8	0.008	0.02	1.3285	1.3416	0.0131	1.0
	0.8	0.008	0.08	1.2821	1.3416	0.0595	4.6
0.75	0.2	0.002	0.02	2.4797	2.4918	0.0121	0.5
	0.2	0.002	0.08	2.4529	2.4918	0.0389	1.6
	0.2	0.008	0.02	2.4964	2.4918	−0.0046	−0.2
	0.2	0.008	0.08	2.4697	2.4918	0.0221	0.9
	0.8	0.002	0.02	2.4189	2.4918	0.0729	3.0
	0.8	0.002	0.08	2.1935	2.4918	0.2983	13.6
	0.8	0.008	0.02	2.4358	2.4918	0.0560	2.3
	0.8	0.008	0.08	2.2108	2.4918	0.2810	12.7

Note that $P_s = 15$, $X = 20$, $\sigma_s = 0.3$, $call_O$ = call price for our model, $call_{BS}$ = call price for B–S model, $\Delta = call_{BS} - call_O$, $\Delta\% = 100\Delta/call_O$, and τ = time to maturity (in year).

when $\tau = 0.25$, $\rho = 0.2$, $\alpha = 0.002$, $\sigma = 0.02$, $call_O = 0.0345$ and $call_{BS} = 0.0351$, which results in a dollar difference $\Delta = call_{BS} - call_O = 0.0006$ and a percentage difference $\Delta\% = 100\Delta/call_O = 1.7$. This overvaluation phenomenon is more pronounced for larger values of τ, ρ, or σ. For example, when $\tau = 0.75$, $\rho = 0.8$, $\alpha = 0.002$, $\sigma = 0.08$, $call_O = 0.2550$ and $call_{BS} = 0.4128$. That is, $\Delta = 0.1578$ and $\Delta\% = 61.9$.

Table 2 shows the two prices for at-the-money ($P_s = 20 = X = 20$) calls. Except for one case, the B–S model moderately overvalues at-the-money calls. For example, when $\tau = 0.75$, $\rho = 0.2$, $\alpha = 0.002$, $\sigma = 0.02$, $call_O = 2.4797$ and $call_{BS} = 2.4918$. That is, $\Delta = 0.0121$ and $\Delta\% = 0.5$. This overvaluation phenomenon becomes more evident for larger τ, ρ, or σ. Table 3 shows the two prices for in-the-money ($P_s = 25 > X = 20$) calls. Except for a few cases, the B–S model slightly overvalues in-the-money calls. For example, when $\tau = 0.25$, $\rho = 0.8$, $\alpha = 0.002$, and $\sigma = 0.02$, $call_O = 5.3735$ and $call_{BS} = 5.3773$. That is, $\Delta = 0.0038$ and $\Delta\% = 0.1$.

Table 3
Prices for in-the-money calls when the short rate at initial time t is 0.06.

τ	ρ	α	σ	$call_O$	$call_{BS}$	Δ	$\Delta\%$
0.25	0.2	0.002	0.02	5.3772	5.3773	0.0001	0.0
	0.2	0.002	0.08	5.3740	5.3773	0.0033	0.1
	0.2	0.008	0.02	5.3806	5.3773	−0.0033	−0.1
	0.2	0.008	0.08	5.3774	5.3773	−0.0001	−0.0
	0.8	0.002	0.02	5.3735	5.3773	0.0038	0.1
	0.8	0.002	0.08	5.3597	5.3773	0.0176	0.3
	0.8	0.008	0.02	5.3770	5.3773	0.0003	0.0
	0.8	0.008	0.08	5.3632	5.3773	0.0141	0.3
0.75	0.2	0.002	0.02	6.3195	6.3227	0.0032	0.1
	0.2	0.002	0.08	6.2981	6.3227	0.0246	0.4
	0.2	0.008	0.02	6.3460	6.3227	−0.0233	−0.4
	0.2	0.008	0.08	6.3246	6.3227	−0.0019	−0.0
	0.8	0.002	0.02	6.2801	6.3227	0.0426	0.7
	0.8	0.002	0.08	6.1423	6.3227	0.1804	2.9
	0.8	0.008	0.02	6.3069	6.3227	0.0158	0.3
	0.8	0.008	0.08	6.1702	6.3227	0.1525	2.5

Note that $P_s = 15$, $X = 20$, $\sigma_s = 0.3$, $call_O$ = call price for our model, $call_{BS}$ = call price for B–S model, $\Delta = call_{BS} - call_O$, $\Delta\% = 100\Delta/call_O$, and τ = time to maturity (in year).

4. Conclusion

Previous option pricing studies typically assume that the short rate is constant over the life of the option. In reality, the short rate is evolving randomly through time. This study derives explicit pricing formulas for European call and put on a stock when the short rate follows the Merton model. Our analysis indicates that each parameter in our option valuation model exerts different degree of effect on the call price. In addition, using our model as a benchmark, our analysis suggests that, in general, the B–S model overvalues out-of-the-money calls, moderately overvalues at-the-money calls, and slightly overvalues in-the-money calls. Our analysis is directly extensible to American calls on non-dividend-paying stocks and to European puts by virtue of put-call parity.

Acknowledgements

We are grateful to Professor Robert Beauwens and an anonymous reviewer for their valuable comments which have improved substantially our paper. We would like to thank Andrew Carverhill and Alan Wong for helpful comments and suggestions.

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